Numerical Methods Assignment 2

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1. Gaussian Elimination with Partial Pivoting (15%)

Solve the following system by Gaussian elimination with partial pivoting:

$$\begin{pmatrix} 4 & 2 & -2 & -1 \\ 0 & 4 & 1 & 2 \\ 3 & -2 & 1 & 2 \\ 2 & 0 & 3 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7 \\ 10 \\ 2 \\ 3 \end{pmatrix}$$

Solution Process

Initial augmented matrix:

$$\left(\begin{array}{cccc|c} 4 & 2 & -2 & -1 & 7 \\ 0 & 4 & 1 & 2 & 10 \\ 3 & -2 & 1 & 2 & 2 \\ 2 & 0 & 3 & -5 & 3 \end{array}\right)$$

Step 1 - Column 1: The pivot element is $a_{11} = 4$. Checking column 1: $|4| \geq |0|, |3|, |2|$. No row interchange needed.

Multipliers: $m_{21} = 0$, $m_{31} = \frac{3}{4} = 0.75$, $m_{41} = \frac{2}{4} = 0.5$

After elimination:

$$\left(\begin{array}{cccc|c} 4 & 2 & -2 & -1 & 7 \\ 0 & 4 & 1 & 2 & 10 \\ 0 & -3.5 & 2.5 & 2.75 & -3.25 \\ 0 & -1 & 4 & -4.5 & -0.5 \end{array}\right)$$

Step 2 - Column 2: The pivot element is $a_{22} = 4$. Checking: $|4| \geq |-3.5|, |-1|$. No row interchange needed.

Multipliers: $m_{32} = -\frac{3.5}{4} = -0.875$, $m_{42} = -\frac{1}{4} = -0.25$

After elimination:

$$\left(\begin{array}{cccc|c} 4 & 2 & -2 & -1 & 7 \\ 0 & 4 & 1 & 2 & 10 \\ 0 & 0 & 3.375 & 4.5 & 5.5 \\ 0 & 0 & 4.25 & -4 & 2 \end{array}\right)$$

Step 3 - Column 3: Checking: $|3.375| < |4.25|$. **Row interchange required: swap row 3 with row 4.**

After interchange:

$$\left(\begin{array}{cccc|c} 4 & 2 & -2 & -1 & 7 \\ 0 & 4 & 1 & 2 & 10 \\ 0 & 0 & 4.25 & -4 & 2 \\ 0 & 0 & 3.375 & 4.5 & 5.5 \end{array}\right)$$

Multiplier: $m_{43} = \frac{3.375}{4.25} = 0.7941$

After elimination:

$$\left(\begin{array}{cccc|c} 4 & 2 & -2 & -1 & 7 \\ 0 & 4 & 1 & 2 & 10 \\ 0 & 0 & 4.25 & -4 & 2 \\ 0 & 0 & 0 & 7.6765 & 3.9118 \end{array}\right)$$

Back substitution:

- $x_4 = \frac{3.9118}{7.6765} = 0.509579$
- $x_3 = \frac{2-(-4)(0.509579)}{4.25} = 0.950192$
- $x_2 = \frac{10-(1)(0.950192)-(2)(0.509579)}{4} = 2.007663$
- $x_1 = \frac{7-(2)(2.007663)-(-2)(0.950192)-(-1)(0.509579)}{4} = 1.348659$

Solution

$$x_1 = 1.348659, \quad x_2 = 2.007663, \quad x_3 = 0.950192, \quad x_4 = 0.509579$$

Column requiring row interchange: Column 3

2. Scaled Partial Pivoting (25%)

Solve the system using scaled partial pivoting:

$$\begin{pmatrix} 4.13 & -2.20 & 0.95 & 3.02 \\ 6.14 & 4.45 & -1.45 & -4.02 \\ 1.03 & 1.86 & 0.44 & 5.22 \end{pmatrix}$$

Part (a): Six Significant Digits

Scale factors:

- $s_1 = \max(|4.13|, |-2.20|, |0.95|) = 4.13$
- $s_2 = \max(|6.14|, |4.45|, |-1.45|) = 6.14$
- $s_3 = \max(|1.03|, |1.86|, |0.44|) = 1.86$

Step 1 - Column 1:

Scaled ratios:

- Row 1: $|4.13\frac{|}{4.13} = 1.000$
- Row 2: $|6.14\frac{|}{6.14} = 1.000$
- Row 3: $|1.03\frac{|}{1.86} = 0.554$

Rows 1 and 2 are tied; keep current order.

Multipliers: $m_{21} = \frac{6.14}{4.13} = 1.48668$, $m_{31} = \frac{1.03}{4.13} = 0.249395$

After elimination:

$$\left(\begin{array}{ccc|c} 4.13 & -2.20 & 0.95 & 3.02 \\ 0 & 7.7207 & -2.86235 & -8.50977 \\ 0 & 2.40867 & 0.203075 & 4.46683 \end{array}\right)$$

Step 2 - Column 2:

Scaled ratios:

- Row 2: $|7.7207\frac{|}{6.14} = 1.257$
- Row 3: $|2.40867\frac{|}{1.86} = 1.295$

Row 3 has larger ratio. **Interchange rows 2 and 3.**

Multiplier: $m_{22} = \frac{7.7207}{2.40867} = 3.20538$

After elimination:

$$\left(\begin{array}{ccc|c} 4.13 & -2.20 & 0.95 & 3.02 \\ 0 & 2.40867 & 0.203075 & 4.46683 \\ 0 & 0 & -3.51328 & -22.8277 \end{array}\right)$$

Back substitution:

- $x_3 = -\frac{22.8277}{-3.51328} = 6.49755$
- $x_2 = \frac{4.46683 - 0.203075 \times 6.49755}{2.40867} = 1.30667$
- $x_1 = \frac{3.02 - 0.95 \times 6.49755 - (-2.20) \times 1.30667}{4.13} = -0.0673$

Solution (6 significant digits):

$$x_1 = -0.0673, \quad x_2 = 1.3067, \quad x_3 = 6.4976$$

Part (b): Three Significant Digits

Following the same process with 3-digit arithmetic:

Solution (3 significant digits):

$$x_1 = -0.0688, \quad x_2 = 1.31, \quad x_3 = 6.51$$

Comparison

The solutions are close but show small differences due to rounding errors:

- x_1: -0.0673 vs -0.0688 (difference $\approx 2\%$)
- x_2: 1.3067 vs 1.31 (difference $\approx 0.3\%$)
- x_3: 6.4976 vs 6.51 (difference $\approx 0.2\%$)

The solution with 3 significant digits is slightly different but still reasonable. The accumulated rounding errors affect x_1 the most since it is computed last in back substitution.

3. Symmetric Tridiagonal System (30%)

The system:

$$\begin{pmatrix} 4 & -1 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 100 \\ 200 \\ 200 \\ 200 \\ 200 \\ 100 \end{pmatrix}$$

Part (a): Algorithm for Symmetric Tridiagonal System

For a symmetric tridiagonal matrix, we only need to store:

- d_i: diagonal elements (n elements)
- a_i: off-diagonal elements ($n - 1$ elements, same above and below diagonal)
- b_i: right-hand side (n elements)

Algorithm (Modified Thomas Algorithm for Symmetric Case):

```
// Input: d[1..n] (diagonal), a[1..n-1] (off-diagonal), b[1..n] (RHS)
// Output: x[1..n] (solution)

// Forward Elimination
for i = 2 to n:
    m = a[i-1] / d[i-1]
    d[i] = d[i] - m * a[i-1]
    b[i] = b[i] - m * b[i-1]

// Back Substitution
x[n] = b[n] / d[n]
for i = n-1 downto 1:
    x[i] = (b[i] - a[i] * x[i+1]) / d[i]
```

Part (b): Solving the System

Initial compact representation:

Row	Sub-diagonal	Diagonal	RHS
1	-	4	100
2	−1	4	200
3	−1	4	200
4	−1	4	200
5	−1	4	200
6	−1	4	100

Forward Elimination:

- Step 1: $m = -\frac{1}{4} = -0.25$, $d_2' = 4 - (-0.25)(-1) = 3.75$, $b_2' = 200 - (-0.25)(100) = 225$
- Step 2: $m = -\frac{1}{3.75} = -0.2667$, $d_3' = 3.7333$, $b_3' = 260$
- Step 3: $m = -\frac{1}{3.7333} = -0.2679$, $d_4' = 3.7321$, $b_4' = 269.64$
- Step 4: $m = -\frac{1}{3.7321} = -0.2679$, $d_5' = 3.7321$, $b_5' = 272.25$
- Step 5: $m = -\frac{1}{3.7321} = -0.2679$, $d_6' = 3.7321$, $b_6' = 172.95$

Back Substitution:

- $x_6 = \frac{172.95}{3.7321} = 46.34$
- $x_5 = \frac{272.25-(-1)(46.34)}{3.7321} = 85.37$
- $x_4 = \frac{269.64-(-1)(85.37)}{3.7321} = 95.12$
- $x_3 = \frac{260-(-1)(95.12)}{3.7333} = 95.12$
- $x_2 = \frac{225-(-1)(95.12)}{3.75} = 85.37$
- $x_1 = \frac{100-(-1)(85.37)}{4} = 46.34$

Solution:

$$x_1 = 46.34, \quad x_2 = 85.37, \quad x_3 = 95.12, \quad x_4 = 95.12, \quad x_5 = 85.37, \quad x_6 = 46.34$$

Part (c): Arithmetic Operations Count

For a system of N equations:

Forward Elimination (loop runs $N - 1$ times):

- $N - 1$ divisions (computing m)
- $N - 1$ multiplications (computing $m \cdot a_{i-1}$)
- $2(N - 1)$ subtractions (updating d_i and b_i)

Total: $4(N - 1)$ operations

Back Substitution:

- 1 division (computing x_N)
- For $i = N - 1$ down to 1: 1 multiplication + 1 subtraction + 1 division = $3(N - 1)$ operations

Total: $1 + 3(N - 1) = 3N - 2$ operations

Total operations: $4(N - 1) + 3N - 2 = 7N - 6 = O(N)$

This is much more efficient than general Gaussian elimination which requires $O(N^3)$ operations.

4. Jacobi and Gauss-Seidel Methods (40%)

Original system:

$$\begin{pmatrix} 7 & -3 & 4 \\ -3 & 2 & 6 \\ 2 & 5 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -5 \end{pmatrix}$$

Checking and Rearranging for Diagonal Dominance

Original matrix diagonal dominance check:

- Row 1: $|7| = 7$, $|-3| + |4| = 7$ (equal, not strictly dominant)
- Row 2: $|2| = 2$, $|-3| + |6| = 9$ (NOT dominant)
- Row 3: $|3| = 3$, $|2| + |5| = 7$ (NOT dominant)

Rearrangement: Swap equations 2 and 3:

$$\begin{pmatrix} 7 & -3 & 4 \\ 2 & 5 & 3 \\ -3 & 2 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \\ 2 \end{pmatrix}$$

After rearrangement:

- Row 1: $|7| = 7$, $|-3| + |4| = 7$ (weakly dominant)
- Row 2: $|5| = 5$, $|2| + |3| = 5$ (weakly dominant)
- Row 3: $|6| = 6 > |-3| + |2| = 5$ (strictly dominant)

The system is weakly diagonally dominant. The methods may still converge.

Part (a): Jacobi Method

Iteration formulas:

$$x_1^{(k+1)} = \frac{6 + 3x_2^{(k)} - 4x_3^{(k)}}{7}$$
$$x_2^{(k+1)} = \frac{-5 - 2x_1^{(k)} - 3x_3^{(k)}}{5}$$
$$x_3^{(k+1)} = \frac{2 + 3x_1^{(k)} - 2x_2^{(k)}}{6}$$

Starting vector: $\boldsymbol{x}^{(0)} = [0, 0, 0]^T$

Iteration	x_1	x_2	x_3
0	0	0	0
1	0.8571	−1.0000	0.3333
2	0.2381	−1.5429	1.0952
3	−0.4299	−1.7524	0.9667
4	−0.4463	−1.4080	0.7025
5	−0.1477	−1.2430	0.5795
...	...	...	...
37	−0.1433	−1.3746	0.7199

Jacobi converged after 37 iterations.

Solution: $x_1 = -0.1433, \quad x_2 = -1.3746, \quad x_3 = 0.7199$

Part (b): Gauss-Seidel Method

Iteration formulas (using most recent values):

$$x_1^{(k+1)} = \frac{6 + 3x_2^{(k)} - 4x_3^{(k)}}{7}$$
$$x_2^{(k+1)} = \frac{-5 - 2x_1^{(k+1)} - 3x_3^{(k)}}{5}$$
$$x_3^{(k+1)} = \frac{2 + 3x_1^{(k+1)} - 2x_2^{(k+1)}}{6}$$

Iteration	x_1	x_2	x_3
0	0	0	0
1	0.8571	−1.3429	1.2095
2	−0.4095	−1.5619	0.6492
3	−0.1832	−1.3162	0.6805
4	−0.0958	−1.3700	0.7421
5	−0.1540	−1.3836	0.7175
…	…	…	…
16	−0.1433	−1.3746	0.7199

Gauss-Seidel converged after 16 iterations.

Solution: $x_1 = -0.1433, \quad x_2 = -1.3746, \quad x_3 = 0.7199$

Comparison

Both methods converge to the same solution:

$$x_1 \approx -0.143, \quad x_2 \approx -1.375, \quad x_3 \approx 0.720$$

Gauss-Seidel converged in **16 iterations**, while Jacobi required **37 iterations**. This is expected because Gauss-Seidel uses updated values immediately, leading to faster convergence.